Continuity, Discontinuity and Intermediate Value Theorem

I Continuity

The precise definition of continuity is simple:

If $f(x)$ is continuous at poit a, then

$$\lim_{x \to a} f(x) = f(x)$$

However, this infers 3 conditions that must be true:

1). $f(a)$ exists

2). $\lim_{x \to a} f(x)$ exists

3). $f(a) = \lim_{x \to a} f(x)$

When you're asked to evaluate the continuity at a certain point, always check these 3 conditions.

ex. Is function $f(x) = \begin{cases} x+1 & x<2 \\ 2x-1 & x \geq 2 \end{cases}$ Continuous at $x=2$?

Answer: 1). $f(2) = 3$ ($f(2)$ exists)

2). $\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2^-} f(x) = 3$

$\Rightarrow \lim_{x \to 2} f(x) = 3$ (exists)

3). $f(2) = \lim_{x \to 2} f(x) = 3$

The function is continuous at $x=2$

- Continuity at an interval

$f(x)$ is continuous in an interval only if it is continuous at all points in that interval.

All functions (polynomial, power, rational, trig, exponential) are continuous in their domains.

II. Discontinuities

3 Types of Discontinuities:

1. Removable
2. Infinite (Essential)
3. Jump

- Removable

1). $\lim_{x \to a} f(x)$ exists

2). $f(a)$ does not exist unless specified

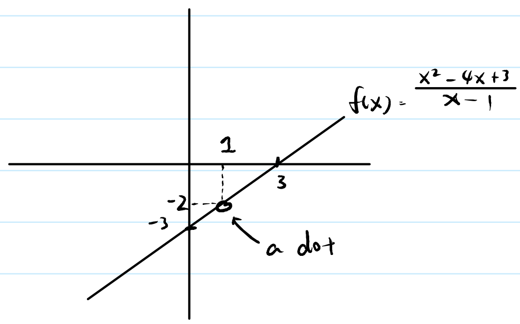
3). If $f(a)$ exists

$$\lim_{x \to a} f(x) \neq f(a)$$

Removable discontinuities often occur when a rational function can be simplified by factoring.

ex. $$f(x) = \frac{x^2 - 4x + 3}{x - 1}$$

has a removable discontinuity at $x = 1$

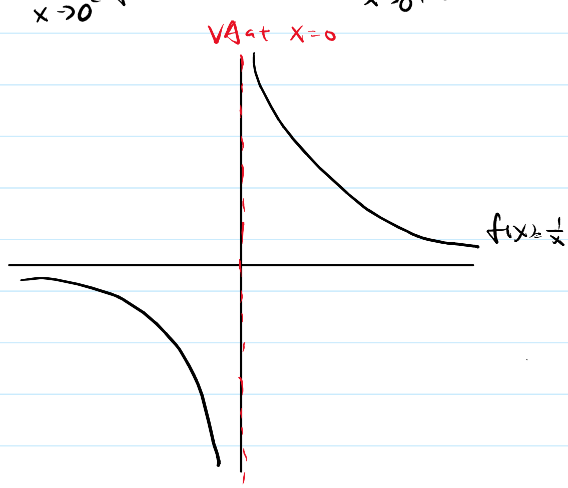


- Infinite (Essential) Discontinuities

When $\lim_{x \to a} f(x) = \pm\infty$ (A vertical asymptote exists)

ex. $f(x) = \frac{1}{x}$ has an infinite discontinuity at $x = 0$

$$\lim_{x \to 0^-} f(x) = -\infty \qquad \lim_{x \to 0^+} f(x) = +\infty$$

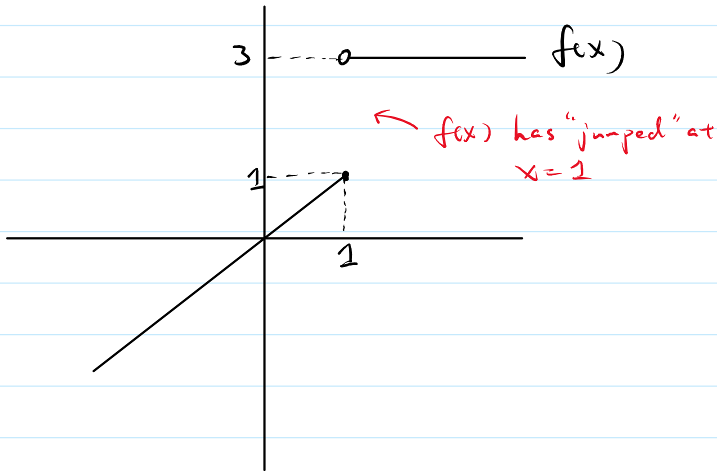


- Jump Discontinuity

When $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

ex. $f(x) = \begin{cases} x & x \le 1 \\ 3 & x > 1 \end{cases}$ has a jump discontinuity at $x = 1$

$\lim_{x \to 1^-} f(x) = 1 \neq \lim_{x \to 1^+} f(x) = 3$



III. The Intermediate Value Theorem (IVT)

If $f(x)$ is continuous on the <u>closed</u> interval $[a, b]$, and N is a number between $f(a)$ and $f(b)$, then it guarantees there exists at least one value of x such that $f(x) = N$. (Given that $a \neq b$)

Comment: the IVT is a theorem that merely <u>guarantees</u> something if conditions are met.

ex. Show that there exists at least one root for equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Answer: If a root to the equation exists, it means there exists at least one value of x such that

$$f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$$

To use the IVT, $a=1$, $b=2$ and $N=0$

Check the conditions of IVT:

1). $a \neq b$

2). since $f(x)$ is a polynomial, it is continuous everywhere

$f(1) = -1 \quad < 0$

$f(2) = 12 \quad > 0$

Then, by IVT, there exists at least one x such that $f(x) = N$